A short note on two-positive Ricci curvature and positive Einstein curvature

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July 10, 2008

Abstract

In this short note we discuss some recent results about two-positive Ricci curvature and their applications to positive Einstein curvature.

1 Surgeries and the fundamental group

We start with the following general fact, see [2] were this fact was used to prove that positive isotropic curvature cannot be stable under 1-dimensional surgeries.

Proposition 1.1 *Let $n \geq 4$, suppose a geometric property P_n on compact n-dimensional manifolds is satisfied by $S^{n-1} \times S^1$: the product of a sphere of dimension $n-1$ with a circle. Assume also that the property P_n is preserved under surgeries in codimensions $\geq n - 1$, then any finitely presented group can be realized as the fundamental group of a compact n-manifold satisfying the property P_n.*

Proof. Taking enough connected sums (surgeries of codimension n) of $S^{n-1} \times S^1$ one can get a compact manifold with the desired set of generators and with the property P_n, then one uses surgeries of codimension $n - 1$ along embeded circles to create the desired relations. The result is a compact n dimensional manifold with the desired fundamental group and satisfies the property P_n. $\diamond$

Consequently, the following geometric properties do not impose any restrictions on the fundamental group:

- Positive scalar curvature in dimension $n \geq 4$.

- Two-positive Ricci curvature in dimension $n \geq 5$.

- Positive Einstein curvature in dimension $n \geq 5$.

2   Two-positive Ricci curvature

The k-Ricci curvature of a Riemannian manifold (M, g) is the function defined on the k-Grassmanian bundle of M which associates to a tangent k-plane P the average of the Ricci curvature in this plane that is $\sum_{i=1}^{k} Ric(e_i, e_i)$ where $\{e_1, ..., e_k\}$ is an arbitrary orthonormal basis of P, see page 311 in [1].

In a recent work Wolfson proved, using the complex form of the second variation of minimal surfaces, the following interesting result about 2-Ricci curvature in four dimensions:

Theorem 2.1 ([4]) *Let M be a closed 4-dimensional Riemannian manifold with positive 2-Ricci curvature then the fundamental group of M must be virtually free.*

Wolfson asked the question whether this property remains true for higher dimensions. It turns out that this property is no longer true in higher dimensions as explained below.

Let us first mention that the stability of positive k-Ricci curvature under surgeries was studied in [1], (see the appendix in page 311), where they were denoted by k_p. For example the corresponding result for $p = 2$ that is for two-positive Ricci curvature is

Theorem 2.2 ([1]) *On a compact n-dimensional manifold, two-positive Ricci curvature is preserved by surgeries of codimension $\geq \max\{4, n-1\}$.*

In particular in dimension 4, 2-positive Ricci curvature is stable under surgeries of codimension 4, that is a connected sum, but generally it is not stable under a surgery of codimension 3 as explained in the first section. Furthermore, in dimensions higher than five we have stability in codimensions n and $n - 1$. This shows that in dimensions ≥ 5, one cannot get restrictions on the fundamental group of a compact manifold with two-positive Ricci curvature. Precisely we have

Corollary 2.3 *Let $n \geq 5$. Any finitely presented group can be made as the fundamental group of a compact manifold of dimension n with 2-positive Ricci curvature.*

3   Positive Einstein curvature

Positive Einstein curvature [3] is equivalent to the assumption that the Einstein tensor is positive definite. It is related to the positivity of the k-Ricci curvature in the following way:

Proposition 3.1 *On a Riemannian n-manifold positive Einstein curvature implies the positivity of the $(n-2)$-Ricci curvature. In particular, in four dimensions positive Einstein curvature implies two-positive Ricci curvature.*

Proof. The positivity of the Einstein curvature implies that $Scal > 2Ric(e_i, e_i)$ for any tangent vector e_i. Therefore we get

$$Scal > Ric(e_i, e_i) + Ric(e_j, e_j),$$

for any two orthonormal tangent vectors e_i, e_j. This last condition is equivalent to $(n-2)$-positive Ricci curvature. $\diamond$

In [3] we asked the following question:

Does the product of a two dimensional sphere with a k dimensional torus admits a Riemannian metric with positive Einstein curvature?

This is trivial in the case $k = 1$ as in dimension 3, positive Einstein curvature is equivalent to positive sectional curvature. Using the results of Wolfson [4] and the above proposition one gets easily a positive answer to this question in the case $k = 2$, as follows:

Proposition 3.2 *The product of a two-dimensional sphere with a two-dimensional torus does not admit any Riemannian metric with positive Einstein curvature.*

In fact, the same is true for the product of a sphere with a Riemann surface with genus ≥ 1. The previous proposition provides us with a first example of a compact manifold of dimension ≥ 4 which admits a metric with positive scalar curvature but does not admit any metric with positive definite Einstein tensor.

The question above remains open for $k \geq 3$.

References

[1] Labbi M.L., Stability of the p-curvature positivity under surgeries and manifolds with positive Einstein tensor, Annals of Global analysis and geometry, 15: 299-312, 1997.

[2] Labbi M. L., On positive isotropic curvature and surgeries, Differential Geometry and its applications, 17, 37-42, 2002.

[3] Labbi M. L., Compact manifolds with positive Einstein curvature, Geometriae Dedicata, 108, 205-217, 2004.

[4] Wolfson J., Four manifolds with two-positive Ricci curvature, preprint, arXiv0805.4183v2.